

On Pythagoras theorem for products of spectral triples

Francesco D'Andrea¹ and Pierre Martinetti^{2,3}

¹ Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Napoli, Italy.

² Dipartimento di Matematica e CMTP, Università di Roma Tor Vergata, Roma, Italy.

³ Dipartimento di Fisica, Università di Roma Sapienza, Roma, Italy.

Abstract

We discuss a version of Pythagoras theorem in noncommutative geometry. Usual Pythagoras theorem can be formulated in terms of Connes' distance, between pure states, in the product of commutative spectral triples. We investigate the generalization to both non pure states and arbitrary spectral triples. We show that Pythagoras theorem is replaced by some Pythagoras inequalities, that we prove for the product of arbitrary (i.e. non-necessarily commutative) spectral triples, assuming only some unitality condition. We show that these inequalities are optimal, and provide non-unital counter-examples inspired by K-homology.

1 Introduction

Given the natural spectral triple $(\mathcal{A}, \mathcal{H}, D)$ associated to a complete Riemannian spin manifold M without boundary, that is

$$\mathcal{A} = C_0^\infty(M), \quad \mathcal{H} = L_2(M, S), \quad D = \not{D}, \quad (1)$$

the spectral distance $d_{\mathcal{A}, D}$ of Connes (see §2.1) on the state space $\mathcal{S}(\mathcal{A})$ of \mathcal{A} coincides with the Wasserstein distance W of order 1 in the theory of optimal transport with cost function the geodesic distance d_{geo} . Namely, given two probability measures μ, μ' on M viewed as states φ, φ' of \mathcal{A} , that is

$$\varphi(f) = \int_M f(x) d\mu \quad \forall f \in \mathcal{A}$$

and similarly for φ', μ' , one has

$$d_{\mathcal{A}, D}(\varphi, \varphi') = W(\mu, \mu') \quad \forall \varphi, \varphi' \in \mathcal{S}(\mathcal{A}). \quad (2)$$

MSC-class 2010: 58B34 (Primary), 46L87 (Secondary).

Keywords: Noncommutative geometry, spectral triples, spectral distance, Pythagoras theorem.

Acknowledgments. P.M. is supported by the ERG-Marie Curie fellowship 237927 “NCG and quantum gravity” and the ERC Advanced Grant 227458 OACFT “Operator Algebras and Conformal Field Theory”.

On the space of pure states $\mathcal{P}(\mathcal{A}) \simeq M$, that by Gelfand duality are Dirac's delta distributions $\delta_x(f) := f(x) \forall x \in M, f \in C(M)$, the spectral/Wasserstein distance gives back the cost function

$$d_{\mathcal{A},D}(x, y) = W(\delta_x, \delta_y) = d_{\text{geo}}(x, y). \quad (3)$$

Consider now the product of two manifolds M_1, M_2 equipped with the product metric. Pythagoras theorem states that

$$d_{\text{geo}}(x, x') = \sqrt{d_{\text{geo}}(x_1, x'_1)^2 + d_{\text{geo}}(x_2, x'_2)^2}, \quad (4)$$

for any couple of points $x = (x_1, x_2)$ and $x' = (x'_1, x'_2) \in M_1 \times M_2$. Denoting $(\mathcal{A}, \mathcal{H}, D)$ the product of the spectral triples of M_1 and M_2 , eq. (4) can be equivalently formulated in terms of spectral distances as

$$d_{\mathcal{A},D}(\delta_{x_1} \otimes \delta_{x_2}, \delta_{x'_1} \otimes \delta_{x'_2}) = \sqrt{d_{\mathcal{A}_1,D_1}(\delta_{x_1}, \delta_{x'_1})^2 + d_{\mathcal{A}_2,D_2}(\delta_{x_2}, \delta_{x'_2})^2} \quad (5)$$

for any pairs of pure states $\delta_{x_1} \otimes \delta_{x_2}, \delta_{x'_1} \otimes \delta_{x'_2}$ in $\mathcal{P}(\mathcal{A})$. In other terms, the product of two manifolds (in the sense of spectral triple) is orthogonal (in the sense of Pythagoras theorem restricted to the pure state space).

It is known for many years [21] that a similar result holds in the discrete case, that is for the product of a manifold by \mathbb{C}^2 , as well as for the product of a manifold by the finite dimensional algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ describing the internal degrees of freedom of the standard model of elementary particles [2]. Furthermore, in the last case Pythagoras theorem yields a metric interpretation of the Higgs field. Recently, it comes out in [20] that eq. (5) for is also true for the product of the Moyal plane by \mathbb{C}^2 , but only between translated states, that is for $\delta_{x'_1}, \delta_{x'_2}$ the two pure states of \mathbb{C}^2 , $\delta_{x_1} = \varphi$ any state of the Moyal algebra and $\delta_{x_2} = \varphi \circ \tau_\kappa$ with $\tau_\kappa, \kappa \in \mathbb{R}^2$, the translation action of \mathbb{R}^2 on the Moyal plane. For arbitrary two states of the Moyal algebra, it is not known whether Pythagoras equality is still valid: a crucial tool of the proof that is missing in the general case is the existence of a geodesic between the states under consideration (as the Riemannian geodesic on the manifold, and the orbit of the translation group on the Moyal plane).

In this paper, we investigate the generalization of Pythagoras theorem to both non-pure states and the product $(\mathcal{A}, \mathcal{H}, D)$ of arbitrary spectral triples $(\mathcal{A}_i, \mathcal{H}_i, D_i)$, $i = 1, 2$. We only impose two limitations: separable states, that is

$$\mathcal{S}(\mathcal{A}) \ni \varphi := \varphi_1 \otimes \varphi_2, \quad \varphi_i \in \mathcal{S}(\mathcal{A}_i)$$

and similarly for φ' , and unital spectral triples. The restriction to separable states is natural with respect to the commutative case, and is also discussed on some physical ground in [19, §2.2]. The restriction to unital spectral triples emerges from the computation, and is discussed in the last part of this paper.

Specifically, we show that the following Pythagoras inequalities hold true on separable (non-necessarily pure) states:

$$d_{\mathcal{A},D}(\varphi, \varphi') \geq \sqrt{d_{\mathcal{A}_1,D_1}(\varphi_1, \varphi'_1)^2 + d_{\mathcal{A}_2,D_2}(\varphi_2, \varphi'_2)^2}, \quad (6a)$$

$$d_{\mathcal{A},D}(\varphi, \varphi') \leq \sqrt{2} \sqrt{d_{\mathcal{A}_1,D_1}(\varphi_1, \varphi'_1)^2 + d_{\mathcal{A}_2,D_2}(\varphi_2, \varphi'_2)^2}. \quad (6b)$$

In the non-unital case, only (6b) holds true. As a corollary, one gets a Pythagoras inequality for the Wasserstein distance between separable states of a product of manifolds:

$$\sqrt{W_1(\mu_1, \mu'_1)^2 + W_2(\mu_2, \mu'_2)^2} \leq W(\mu, \mu') \leq \sqrt{2} \sqrt{W_1(\mu_1, \mu'_1)^2 + W_2(\mu_2, \mu'_2)^2}. \quad (7)$$

Moreover we show that both equations (6) and (7) are optimal, i.e. that the coefficient in (6b) and on the r.h.s. of (7) cannot be less than $\sqrt{2}$, by providing examples where this bound is actually saturated.

It is remarkable that “something” of Pythagoras theorem survives in the most general case. This was not granted at all from the beginning, especially having in mind the well known “inverse Pythagoras relation” satisfied by the Dirac operator in the product of spectral triples (see (25) in the conclusion). The later seems to indicate that Pythagoras equality for the spectral distance may be retrieved only in some very particular cases, like the product of a manifold or the Moyal plane by \mathbb{C}^2 , when this inverse relation can be inverted. It is rather unexpected that inequalities (6) holds true in the general case.

Also, from the point of view of the Wasserstein distance and as far as we can judge from a limited knowledge of the literature, it did not seem so well noticed that Pythagoras theorem does not hold for non pure states.

The paper is organized as follows. In §2 we recall some basic definitions. In §3 we discuss Pythagoras theorem for the product of two manifolds. We start with the geodesic distance (between pure states): for the sake of completeness, we provide a proof using differential geometry in §3.1, and explain the relation with noncommutative geometry and products of spectral triples in §3.2, proving eq.(5). Then, in §3.3 we pass to non necessarily pure states and the Wasserstein distance, and show with a simple counterexample that Pythagoras theorem does not hold, and must be replaced by the inequality (7); we prove that the latter is optimal. In §4 we consider the generalization to arbitrary spectral triples, and prove the inequalities (6): the upper bound for $d_{\mathcal{A}, D}(\varphi, \varphi')$ holds for arbitrary spectral triples, while the lower bound can be proved only for a product of two *unital* spectral triples. In §6 we provide two simple examples of non-unital spectral triples violating the above-mentioned lower bound, after a short digression in §5 to explain their importance in K -homology.

2 Spectral and Wasserstein distances

We use the following notations/conventions. $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on a Hilbert space \mathcal{H} . By a $*$ -algebra \mathcal{A} we shall always mean an associative involutive \mathbb{C} -algebra. A $*$ -representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ is called *non-degenerate* if $\{\pi(a)\psi\}_{a \in \mathcal{A}, \psi \in \mathcal{H}}$ span a dense subspace of \mathcal{H} ; when \mathcal{A} is unital, with unit element e , the representation π is called *unital* if $\pi(e) = 1 \in \mathcal{B}(\mathcal{H})$ is the identity operator.

Note that a representation of a unital $*$ -algebra is non-degenerate if and only if it is unital. Indeed, since e is a projection, $\pi(e)$ is a projection, and from $\pi(a) = \pi(a)\pi(e) \forall a \in \mathcal{A}$ it follows that $\pi(\mathcal{A})\mathcal{H}$ coincides with the range of $\pi(e)$; as a corollary, π is non-degenerate if and only if $\ker \pi(e) = \{0\}$, i.e. if and only if $\pi(e) = 1$.

When π is a faithful representation, and there is no risk of ambiguity, we will identify \mathcal{A} with $\pi(\mathcal{A})$ and omit the representation symbol; if $\exists e \in \mathcal{A}$ but $\pi(e) \neq 1$, we identify \mathcal{A} with $\pi(\mathcal{A})$ and think of it as a non-unital subalgebra of $\mathcal{B}(\mathcal{H})$.

When we talk about *states* of \mathcal{A} we always mean states of the C^* -algebra $\bar{\mathcal{A}}$, closure of $\pi(\mathcal{A})$; the set of all states is denoted by $S(\mathcal{A})$. We denote by $\|\cdot\|_{\mathcal{B}(\mathcal{H})}$ the operator norm of $\mathcal{B}(\mathcal{H})$, by $\|v\|_{\mathcal{H}}^2 = \langle v, v \rangle$ the norm of a vector $v \in \mathcal{H}$, and use the notation $\langle \cdot, \cdot \rangle$ for the inner product, regardless of the Hilbert space we are considering.

2.1 Basics on noncommutative spaces

The core of noncommutative differential geometry is the notion of spectral triple [5, 6], also known as *unbounded Fredholm module* (see e.g. [3]) or *K-cycle* (see e.g. [4]).

Recall that a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is the datum of: i) a separable complex Hilbert space \mathcal{H} , ii) a $*$ -algebra \mathcal{A} with a faithful $*$ -representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ (the representation symbol is usually omitted), iii) a (not-necessarily bounded) self-adjoint operator D on \mathcal{H} such that $[D, a]$ is bounded and $a(1 + D^2)^{-1/2}$ is compact, for all $a \in \mathcal{A}$. The spectral triple is *unital* if \mathcal{A} is a unital algebra and π a unital representation. In the last sections §5 and §6.1 we will consider an example where the algebra is unital but the representation is not: this will be regarded as a non-unital spectral triple. Note that non-unital representations of unital algebras are of fundamental importance in K-homology (see §5).

A spectral triple is *even* if there is a *grading* γ on \mathcal{H} , i.e. a bounded operator satisfying $\gamma = \gamma^*$ and $\gamma^2 = 1$, commuting with any $a \in \mathcal{A}$ and anticommuting with D .

A commutative example is given by $(C_0^\infty(M), L^2(M, S), \not{D})$, where $C_0^\infty(M)$ is the algebra of complex-valued smooth functions vanishing at infinity on a Riemannian spin^c manifold with no boundary, $L^2(M, S)$ is the Hilbert space of square integrable spinors and \not{D} is the Dirac operator. This spectral triple is even if M is even dimensional.

The set $S(\mathcal{A})$ is an extended metric space¹, with distance given by

$$d_{\mathcal{A}, D}(\varphi, \varphi') = \sup_{a=a^* \in \mathcal{A}} \{ \varphi(a) - \varphi'(a) : \|[D, a]\|_{\mathcal{B}(\mathcal{H})} \leq 1 \}$$

for all $\varphi, \varphi' \in S(\mathcal{A})$. This is usually called *Connes' metric* or *spectral distance*. When there is no risk of ambiguity, this distance will be denoted simply by d . It has been introduced in [4], with the supremum on the whole algebra \mathcal{A} . It is routine to show that the supremum can be equivalently searched on selfadjoint elements [17].

Rieffel first noticed in [23] that the spectral distance associated to (1) for compact M coincides with the Wasserstein distance of order 1 between two probability measures μ_1, μ_2 on M (with cost given by the geodesic distance d_{geo}) given by [25]:

$$W(\mu_1, \mu_2) := \sup_{\|f\|_{\text{Lip}} \leq 1, f \in L^1(\mu_1) \cap L^1(\mu_2)} \left(\int_{\mathcal{X}} f d\mu_1 - \int_{\mathcal{X}} f d\mu_2 \right),$$

where the supremum is on all real $\mu_{i=1,2}$ -integrable functions f that are 1-Lipschitz, i.e. such that $|f(x) - f(y)| \leq d_{\text{geo}}(x, y) \forall x, y \in M$. This result remains true for locally compact manifold providing one assumes geodesic completeness (see §2.2 of [10]).

¹An extended metric space is a pair (X, d) where X is a set and $d : X \times X \rightarrow [0, \infty]$ a symmetric map satisfying the triangle inequality and such that $d(x, y) = 0$ iff $x = y$. The only difference with an ordinary metric space is that the value $+\infty$ for the distance is allowed.

2.2 Products of spectral triples

In noncommutative geometry, the Cartesian product of spaces is replaced by the product of spectral triples. Given two spectral triples $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ such that the first one is even, their product $(\mathcal{A}, \mathcal{H}, D)$ is defined as

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2, \quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \quad D = D_1 \otimes 1 + \gamma_1 \otimes D_2.$$

Here the tensor product between algebras is the algebraic tensor product.

For simplicity of notations we will only consider the case when at least one of the two spectral triples is even, but one can define the product of two odd spectral triples as well (see e.g. [24, 11]), and all our results can be extended to this case.

Recall that a state $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ is called *separable* if it is of the form $\varphi = \varphi_1 \otimes \varphi_2$, with φ_i a state on \mathcal{A}_i for $i = 1, 2$. When at least one of the \mathcal{A}_i is commutative, all pure states are separable [18], that is $\mathcal{P}(\mathcal{A}) = \mathcal{P}(\mathcal{A}_1) \times \mathcal{P}(\mathcal{A}_2)$.

3 Products of manifolds

In this section, we first recall how to retrieve Pythagoras theorem for a product $M = M_1 \times M_2$ of manifolds, and interpret this easy result of differential geometry within the spectral distance framework. Then we investigate the Wasserstein distance, showing by examples that one cannot hope to prove inequalities stronger than (7).

3.1 Pythagoras theorem: the differential geometry way

Let (M_1, g_1) , (M_2, g_2) be two connected complete Riemannian manifolds of dimension m_1, m_2 , and let M denote their product $M_1 \times M_2$ equipped with the product metric:

$$g := \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}.$$

The line element ds of M is given by $ds^2 = ds_1^2 + ds_2^2$, with ds_i the line element of M_i , $i = 1, 2$. This infinitesimal version of Pythagoras theorem can be integrated in order to obtain Pythagoras equality.

Proposition 1. *For any $x = (x_1, x_2), x' = (x'_1, x'_2) \in M$,*

$$d(x, x')^2 = d_1(x_1, x'_1)^2 + d_2(x_2, x'_2)^2 \tag{8}$$

where d, d_i denote the geodesic distance on M, M_i respectively, $i = 1, 2$.

Proof. Given a geodesic $c(s) = (c_1(s), c_2(s))$ between x and x' in M parametrized by its proper length s , we first show that the projections $c_i(s)$ on M_i satisfy the equation of the geodesics, then that s is an affine parameter for both curves c_i . Pythagoras theorem follows immediately. Let us compute the Christoffel symbol of M ,

$$\Gamma_{ab}^c = \frac{1}{2} g^{cd} (\partial_a g_{db} + \partial_b g_{da} - \partial_d g_{ab})$$

where g_{ab} , $a, b \in [1, m_1 + m_2]$, denote the components of the metric g of M . Writing $g_{\mu\nu}$, $\mu, \nu \in [1, m_1]$ and $g_{\mu'\nu'}$, $\mu', \nu' \in [m_1 + 1, m_1 + m_2]$ the components of the metrics g_1, g_2 of M_1, M_2 , one has

$$g_{\mu\mu'} = g_{\mu'\mu} = 0, \quad \partial_\mu g_{\mu'\nu'} = \partial_{\mu'} g_{\mu\nu} = 0 \quad \forall \mu, \mu', \nu, \nu'$$

so that for any $c \in [1, m_1 + m_2]$, $\Gamma_{\mu\mu'}^c = \Gamma_{\mu'\mu}^c = \Gamma_{\mu\nu}^{\mu'} = \Gamma_{\mu'\nu'}^\mu = 0$. Therefore the geodesic equation on M :

$$\frac{d^2 c^c}{ds^2} + \Gamma_{ab}^c \frac{dc^a}{ds} \frac{dc^b}{ds} = 0$$

separates into two equations on M_1, M_2 :

$$\frac{d^2 c_1^\alpha}{ds^2} + \Gamma_{\mu\nu}^\alpha \frac{dc_1^\mu}{ds} \frac{dc_1^\nu}{ds} = 0, \quad \frac{d^2 c_2^{\alpha'}}{ds^2} + \Gamma_{\mu'\nu'}^{\alpha'} \frac{dc_2^{\mu'}}{ds} \frac{dc_2^{\nu'}}{ds} = 0. \quad (9)$$

Before claiming that c_i 's are geodesic curves in the M_i 's, one has to check that s is an affine parameter for both curves. To fix the notations, we show it for M_1 , the proof for M_2 being similar. Let s_1 denote the proper length of the curve c_1 in M_1 . Its length l_1 is

$$\int_0^{l_1} \left\| \frac{d}{ds_1} c_1 \right\| ds_1.$$

Under the change of parametrization $t \rightarrow s_1$, equation (9) becomes

$$\frac{d^2 c_1^\alpha}{ds_1^2} + \Gamma_{\mu\nu}^\alpha \frac{dc_1^\mu}{ds_1} \frac{dc_1^\nu}{ds_1} = -\frac{dc_1^\mu}{ds_1} \frac{d^2 s_1}{ds^2}. \quad (10)$$

The vector $\dot{c}_1 := \frac{dc_1^\mu}{ds_1} \partial_\mu$, tangent to c_1 , has constant norm $\sqrt{g_1(\dot{c}_1, \dot{c}_1)} = 1$. Hence, using that the metric is parallel with respect to the covariant derivative $\nabla_{\dot{c}_1}$ (of the Levi-Civita connection) along \dot{c}_1 , that is

$$\frac{d}{ds_1} g_1(\dot{c}_1, \dot{c}_1) = 2g_1(\nabla_{\dot{c}_1} \dot{c}_1, \dot{c}_1) = 0,$$

one obtains from (10) — whose l.h.s. is nothing but $\nabla_{\dot{c}_1} \dot{c}_1$ — that:

$$0 = 2g_1(\nabla_{\dot{c}_1} \dot{c}_1, \dot{c}_1) = -2 \frac{d^2 s_1}{ds^2} g_1(\dot{c}_1, \dot{c}_1) = -2 \frac{d^2 s_1}{ds^2}.$$

Hence $s_1 = a_1 s + b_1$ for some constants a_1, b_1 . Similarly $s_2 = a_2 s + b_2$. This means that s is an affine parameter of both curves c_i , so that the latter are geodesics of M_i .

One can parametrize any of the curves c, c_i by any of the parameters s_i, s . In particular, using

$$\frac{ds_1}{ds} = a_1, \quad \frac{ds_2}{ds} = a_2 \quad \text{so that} \quad \frac{ds_2}{ds_1} = \frac{a_2}{a_1},$$

the length l_2 of c_2 can be written as

$$l_2 = \int_0^{l_1} \frac{ds_2}{ds_1} ds_1 = \frac{a_2}{a_1} \int_0^{l_1} ds_1 = \frac{l_1 a_2}{a_1}.$$

As well, the length l of c is

$$\int_0^{l_1} ds = \int \sqrt{1 + \left(\frac{ds_2}{ds_1}\right)^2} ds_1 = \sqrt{1 + \left(\frac{a_1}{a_2}\right)^2} \int_0^{l_1} ds_1 = l_1 \sqrt{1 + \left(\frac{a_1}{a_2}\right)^2} = \sqrt{l_1^2 + l_2^2},$$

which is nothing but (8). ■

3.2 Pythagoras theorem: the noncommutative geometry way

Besides the natural spectral triple (1), given an orientable Riemannian manifold M without boundary, one can define an even spectral triple:

$$\mathcal{A} = C_0^\infty(M), \quad \mathcal{H} = \Omega^\bullet(M), \quad D = d + d^*, \quad (11)$$

with \mathcal{H} the Hilbert space of square integrable differential forms and D the Hodge-Dirac operator (self-adjoint on a suitable domain). The grading $\gamma\omega := (-1)^k\omega$ on k -form is extended by linearity on \mathcal{H} . This spectral triple is even, even if M is odd-dimensional. We will refer to this as the “canonical spectral triple” of M , and denote it by $CS(M)$.

If M is even-dimensional, there are in fact two possible \mathbb{Z}_2 -gradings on \mathcal{H} , both anti-commuting with D : one is the grading γ above, and the other is the Hodge star operator (whose square is 1 using the phase convention of [14]). Therefore, one has two spectral triples that differ only in the grading (thus giving the same distance): in the former case, one usually calls D the *Hodge-Dirac operator* and the corresponding differential complex is the de Rham complex; in the latter case, the operator D is usually called the *signature operator* and the corresponding differential complex is called the signature complex. The signature operator is the one used in Connes reconstruction formula [7], and it is the one interesting in K-homology and index theory [12]. On the other hand, the de Rham complex is the one which is multiplicative under products, as explained in §3.1 of [12] and recalled below. Working with the Hodge-Dirac operator has the advantage (besides the fact that we don’t need a spin structure), that one can use the product of even-even spectral triples, even if both the manifolds M_1 and M_2 are odd-dimensional.

Let now $M = M_1 \times M_2$ be the product of two orientable Riemannian manifolds M_1, M_2 (with product metric). Identifying $C_0(M)$ with the spatial tensor product $C_0(M_1) \bar{\otimes} C_0(M_2)$ [26, App. T], all pure states of $C_0(M)$ are separable: $\delta_x = \delta_{x_1} \otimes \delta_{x_2} \ \forall x = (x_1, x_2) \in M$.

Proposition 2. *The spectral distance associated to the spectral triple $CS(M)$ coincides with the spectral distance associated to the product of the canonical spectral triples of M_1 and M_2 , that we denote by $CS(M_1) \otimes CS(M_2)$.*

Proof. If ω_1 resp. ω_2 is a differential form on M_1 resp. M_2 (with degree k_1 resp. k_2), there is an obvious identification $\Omega^\bullet(M) \simeq \Omega^\bullet(M_1) \otimes \Omega^\bullet(M_2)$ given by the map $\omega_1 \wedge \omega_2 \rightarrow \omega_1 \otimes \omega_2$ (the linear span of forms $\omega_1 \wedge \omega_2$ is dense in $\Omega^\bullet(M)$), and by the graded Leibniz rule $d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge (d\omega_2)$, that is

$$d = d|_{M_1} \otimes 1 + \gamma_1 \otimes d|_{M_2}.$$

By adjunction, one has a similar relation for d^* , proving that the Hodge-Dirac operator on M is $D = D_1 \otimes 1 + \gamma_1 \otimes D_2$, where D_i is the Hodge-Dirac operator on M_i . Since the degree of $\omega_1 \wedge \omega_2$ is the sum of the degrees of ω_1 and ω_2 , one has also $\gamma = \gamma_1 \otimes \gamma_2$. In other terms, the Dirac operator and chirality of $CS(M)$ are the Dirac operator and grading of the product $CS(M_1) \otimes CS(M_2)$.

However $CS(M)$ is not equal nor unitary equivalent to $CS(M_1) \otimes CS(M_2)$, since the algebraic tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2 = C_0^\infty(M_1) \otimes C_0^\infty(M_2)$ is only dense in the algebra:

$$\mathcal{A} = C_0^\infty(M_1 \times M_2) \simeq C_0^\infty(M_1) \hat{\otimes} C_0^\infty(M_2),$$

where $\hat{\otimes}$ is the projective tensor product of complete locally convex Hausdorff topological algebras [15]. Since $\mathcal{A}_1 \otimes \mathcal{A}_2 \subset \mathcal{A}$, clearly $d_{\mathcal{A}_1 \otimes \mathcal{A}_2, D}(\varphi, \varphi') \leq d_{\mathcal{A}, D}(\varphi, \varphi')$.

In fact, the two distances coincide. By definition of projective tensor product, the topology of \mathcal{A} is given by the uniform convergence of functions together with all their derivatives: every element $f \in \mathcal{A}$ is the limit of a sequence of elements $f_n \in \mathcal{A}_1 \otimes \mathcal{A}_2$ which is convergent in the above-mentioned topology. In particular, since f_n is norm-convergent to f , one has $\varphi(f_n) \rightarrow \varphi(f)$ for any state φ . Moreover, the uniform convergence coincides with the convergence in the sup norm (that is also the operator norm on \mathcal{H}), so $[D, \pi(f_n)]$ is also norm-convergent to $[D, \pi(f)]$. This proves that $d_{\mathcal{A}_1 \otimes \mathcal{A}_2, D}(\varphi, \varphi') = d_{\mathcal{A}, D}(\varphi, \varphi')$. ■

Remark 3. *Up to a completion of the algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$, the spectral triples $CS(M)$ and $CS(M_1) \otimes CS(M_2)$ are equivalent. In particular, the Hodge-Dirac operator of M is related to the Hodge-Dirac operators of M_1 and M_2 by the formula $D = D_1 \otimes 1 + \gamma_1 \otimes D_2$. A similar decomposition for the Dirac's Dirac operator holds for \mathbb{R}^n and flat tori [11], and is believed to be true for arbitrary Riemannian spin manifolds.*

Prop. 2 applied to pure states, together with Prop. 1, shows that the spectral distance associated to $CS(M_1) \otimes CS(M_2)$ is the geodesic distance of $M_1 \times M_2$. In other terms, the product of canonical spectral triples of manifolds is orthogonal in the sense of (5).

3.3 Pythagoras inequalities for the Wasserstein distance

Pythagoras theorem holds true for pure states in the product of commutative spectral triples. There are two possible generalization: non-pure states and noncommutative spectral triples. We show on elementary examples that even in the commutative case, Pythagoras theorem does not hold for non-pure states. Noncommutative examples are investigated in the next section.

Consider the Cartesian product $\mathbb{R} \times \mathbb{R}$ with the standard Euclidean metric, and the states

$$\varphi_\lambda(f) := \lambda f(1) + (1 - \lambda)f(0), \quad (12)$$

with $0 \leq \lambda \leq 1$. Let us denote by W_1 resp. W_2 the Wasserstein distance on the first resp. second factor of $\mathbb{R} \times \mathbb{R}$, and by W the Wasserstein distance on the product.

Proposition 4. *Let $k_\lambda = \lambda + \sqrt{2}(1 - \lambda)$. Then:*

$$W(\varphi_\lambda \otimes \varphi_\lambda, \varphi_0 \otimes \varphi_0) = k_\lambda \sqrt{W_1(\varphi_\lambda, \varphi_0)^2 + W_2(\varphi_\lambda, \varphi_0)^2},$$

for any $0 \leq \lambda \leq 1$. Note that k_λ assumes all possible values in $[1, \sqrt{2}]$.

Proof. As recalled in §2.1, $W_1(\varphi_\lambda, \varphi_0) = W_2(\varphi_\lambda, \varphi_0)$ is the supremum of $\lambda\{f(1) - f(0)\}$ over real 1-Lipschitz functions f on \mathbb{R} . This is equal to λ (the sup is attained on the function $f(x) = x$).

On the other hand, identifying $\sum_i f_i \otimes g_i \in C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$ with the function $h \in C_0(\mathbb{R}^2)$, $h(x_1, x_2) = \sum_i f_i(x)g_i(y)$, one has

$$(\varphi_\lambda \otimes \varphi_\lambda)(h) = \lambda^2 h(1, 1) + \lambda(1 - \lambda)(h(1, 0) + h(0, 1)) + (1 - \lambda)^2 h(0, 0).$$

Therefore $W(\varphi_\lambda \otimes \varphi_\lambda, \varphi_0 \otimes \varphi_0)$ is the supremum of

$$\lambda^2\{h(1,1) - h(0,0)\} + \lambda(1-\lambda)\{h(1,0) - h(0,0)\} + \lambda(1-\lambda)\{h(0,1) - h(0,0)\}, \quad (13)$$

over 1-Lipschitz functions h on \mathbb{R}^2 . From $h(x) - h(y) \leq d_{\text{geo}}(x, y)$ it follows that this is no greater than $\sqrt{2}\lambda^2 + 2\lambda(1-\lambda) = \sqrt{2}\lambda k_\lambda$. The supremum is saturated by the function $h(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$, proving that

$$W(\varphi_\lambda \otimes \varphi_\lambda, \varphi_0 \otimes \varphi_0) = \sqrt{2}\lambda k_\lambda = k_\lambda \sqrt{W_1(\varphi_\lambda, \varphi_0)^2 + W_2(\varphi_\lambda, \varphi_0)^2}. \quad \blacksquare$$

Note that for $\lambda \rightarrow 0^+$, k_λ goes to $\sqrt{2}$ and not to zero, although for $\lambda = 0$, Pythagoras equality is trivially satisfied.

One can show that the same argument works on a torus (with flat metric), providing then an example where the space is a compact one. These examples show that the best one may hope, for arbitrary states and manifolds, is an inequality like (7). In the next section, we prove such an inequality, also holding in the noncommutative case.

4 Pythagoras inequalities for products of spectral triples

In this section we consider a product of arbitrary (not necessarily commutative) spectral triples. We shall use the shorthand notation $d = d_{\mathcal{A}, D}$ and $d_i = d_{\mathcal{A}_i, D_i}$ for $i = 1, 2$. Let us state the main theorem.

Theorem 5. *Given two spectral triples $(\mathcal{A}_i, \mathcal{H}_i, D_i)$, $i = 1, 2$, one has:*

i) For any two separable states $\varphi = \varphi_1 \otimes \varphi_2$ and $\varphi' = \varphi'_1 \otimes \varphi'_2$, we have

$$d(\varphi, \varphi') \leq d_1(\varphi_1, \varphi'_1) + d_2(\varphi_2, \varphi'_2). \quad (14a)$$

ii) Furthermore, if the spectral triples are unital, we also have:

$$d(\varphi, \varphi') \geq \sqrt{d_1(\varphi_1, \varphi'_1)^2 + d_2(\varphi_2, \varphi'_2)^2}. \quad (14b)$$

Notice that from (14a) and the observation that $(a+b)^2 = 2(a^2+b^2) - (a-b)^2 \leq 2(a^2+b^2)$ it follows

$$d(\varphi, \varphi') \leq \sqrt{2} \sqrt{d_1(\varphi_1, \varphi'_1)^2 + d_2(\varphi_2, \varphi'_2)^2}. \quad (15)$$

As an easy corollary, one also retrieves a result of [21]:

Corollary 6. *In the unital case, if $\varphi_2 = \varphi'_2$ we have $d(\varphi, \varphi') = d_1(\varphi_1, \varphi'_1)$, and similarly if $\varphi_1 = \varphi'_1$ we have $d(\varphi, \varphi') = d_2(\varphi_2, \varphi'_2)$.*

Theorem 5 generalizes to arbitrary spectral triples the results of [20], where the first triple was assumed to be unital, and the second one was the canonical spectral triple on \mathbb{C}^2 .

Let us recall that by unital spectral triple we mean that both the conditions $\exists e \in \mathcal{A}$ and $\pi(e) = 1$ are satisfied. The importance of this requirement for (14b) is discussed in §6. If $\exists e \in \mathcal{A}$ but $\pi(e) \neq 1$, we still have a legitimate spectral triple, although non-unital, and we stress that (14a) is still valid in this case.

4.1 Proof of the main theorem

This section is devoted to the proof of theorem 5. We need some preliminary lemmas.

Lemma 7. *For any $x, y \geq 0$,*

$$\sup_{\substack{\alpha, \beta \geq 0 \\ \alpha^2 + \beta^2 \leq 1}} (\alpha x + \beta y) = \sqrt{x^2 + y^2}. \quad (16)$$

Proof. If $(x, y) = (0, 0)$ the statement is trivial. Assuming $(x, y) \neq (0, 0)$, by choosing $(\alpha, \beta) = (x, y)/\sqrt{x^2 + y^2}$ one proves that the left hand side of (16) is greater than or equal to the right hand side. On the other hand, by Cauchy-Bunyakovsky-Schwarz inequality

$$\alpha x + \beta y \leq \sqrt{\alpha^2 + \beta^2} \sqrt{x^2 + y^2} \leq \sqrt{x^2 + y^2}$$

if $\alpha^2 + \beta^2 \leq 1$. This proves that the inequality is actually an equality. \blacksquare

Lemma 8. *For any $a = a_1 \otimes 1 + 1 \otimes a_2$, with $a_i \in \mathcal{A}_i$, we have*

$$\|[D, a]\|_{\mathcal{B}(\mathcal{H})}^2 = \|[D_1, a_1]\|_{\mathcal{B}(\mathcal{H}_1)}^2 + \|[D_2, a_2]\|_{\mathcal{B}(\mathcal{H}_2)}^2.$$

Proof. We have

$$[D, a] = [D_1, a_1] \otimes 1 + \gamma_1 \otimes [D_2, a_2].$$

Call $A = [D_1, a_1] \otimes 1$, $B = 1 \otimes [D_2, a_2]$, $\gamma = \gamma_1 \otimes 1$, and notice that $\|A\|_{\mathcal{B}(\mathcal{H})} = \|[D_1, a_1]\|_{\mathcal{B}(\mathcal{H}_1)}$, $\|B\|_{\mathcal{B}(\mathcal{H})} = \|[D_2, a_2]\|_{\mathcal{B}(\mathcal{H}_2)}$, $\|A + \gamma B\|_{\mathcal{B}(\mathcal{H})} = \|[D, a]\|_{\mathcal{B}(\mathcal{H})}$. The lemma amounts to prove that

$$\|A + \gamma B\|_{\mathcal{B}(\mathcal{H})}^2 = \|A\|_{\mathcal{B}(\mathcal{H})}^2 + \|B\|_{\mathcal{B}(\mathcal{H})}^2.$$

Since $A = -A^*$, $B = -B^*$, $A\gamma + \gamma A = 0$, $B\gamma - \gamma B = 0$ and $[A, B] = 0$, we have

$$(A + \gamma B)^*(A + \gamma B) = -A^2 - B^2 + \gamma[A, B] = -A^2 - B^2,$$

so that, by the triangle inequality and the C^* -norm property,

$$\|A + \gamma B\|_{\mathcal{B}(\mathcal{H})}^2 \leq \|A^2\|_{\mathcal{B}(\mathcal{H})} + \|B^2\|_{\mathcal{B}(\mathcal{H})}.$$

To prove the opposite inequality, let us consider a supremum over vectors $v = v_1 \otimes v_2$:

$$\begin{aligned} \|A + \gamma B\|_{\mathcal{B}(\mathcal{H})}^2 &\geq \sup_{0 \neq v = v_1 \otimes v_2 \in \mathcal{H}} \frac{\langle (A + \gamma B)v, (A + \gamma B)v \rangle}{\|v\|_{\mathcal{H}}^2} = \sup_{0 \neq v = v_1 \otimes v_2 \in \mathcal{H}} \frac{\langle v, -(A^2 + B^2)v \rangle}{\|v\|_{\mathcal{H}}^2} \\ &= \sup_{0 \neq v = v_1 \otimes v_2 \in \mathcal{H}} \left\{ \frac{\langle v, -A^2 v \rangle}{\|v\|_{\mathcal{H}}^2} + \frac{\langle v, -B^2 v \rangle}{\|v\|_{\mathcal{H}}^2} \right\} = \sup_{0 \neq v = v_1 \otimes v_2 \in \mathcal{H}} \left\{ \frac{\langle Av, Av \rangle}{\|v\|_{\mathcal{H}}^2} + \frac{\langle Bv, Bv \rangle}{\|v\|_{\mathcal{H}}^2} \right\} \\ &= \|A\|_{\mathcal{B}(\mathcal{H}_1)}^2 + \|B\|_{\mathcal{B}(\mathcal{H}_2)}^2. \end{aligned}$$

This proves the lemma. \blacksquare

Proof of Theorem 5, point (ii). Let $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ be two unital spectral triples and $\varphi = \varphi_1 \otimes \varphi_2$ and $\varphi' = \varphi'_1 \otimes \varphi'_2$ two separable states. By definition

$$d(\varphi, \varphi') = \sup_{a = a^* \in \mathcal{A}} \left\{ \varphi(a) - \varphi'(a) : \|[D, a]\|_{\mathcal{B}(\mathcal{H})}^2 \leq 1 \right\}.$$

We get a lower bound if we take the supremum over elements of the form $a = a_1 \otimes 1 + 1 \otimes a_2$, with $a_1 = a_1^* \in \mathcal{A}_1$ and $a_2 = a_2^* \in \mathcal{A}_2$. Since

$$\varphi(a) - \varphi'(a) = \varphi_1(a_1) - \varphi'_1(a_1) + \varphi_2(a_2) - \varphi'_2(a_2) ,$$

by Lemma 8 we get

$$\begin{aligned} d(\varphi, \varphi') &\geq \sup_{a_i = a_i^* \in \mathcal{A}_i} \left\{ \varphi_1(a_1) - \varphi'_1(a_1) + \varphi_2(a_2) - \varphi'_2(a_2) : \right. \\ &\quad \left. \|[D_1, a_1]\|_{\mathcal{B}(\mathcal{H}_1)}^2 + \|[D_2, a_2]\|_{\mathcal{B}(\mathcal{H}_2)}^2 \leq 1 \right\} \\ &= \sup_{\alpha^2 + \beta^2 \leq 1} \left\{ \sup_{a_1 = a_1^* \in \mathcal{A}_1} \left\{ \varphi_1(a_1) - \varphi'_1(a_1) : \|[D_1, a_1]\|_{\mathcal{B}(\mathcal{H}_1)} \leq \alpha \right\} \right. \\ &\quad \left. + \sup_{a_2 = a_2^* \in \mathcal{A}_2} \left\{ \varphi_2(a_2) - \varphi'_2(a_2) : \|[D_2, a_2]\|_{\mathcal{B}(\mathcal{H}_2)} \leq \beta \right\} \right\} \\ &= \sup_{\alpha^2 + \beta^2 \leq 1} \left\{ \alpha d_1(\varphi_1, \varphi'_1) + \beta d_2(\varphi_2, \varphi'_2) \right\} . \end{aligned}$$

Applying Lemma 7 to last equation, we prove (14b). ■

Remark: note that what spoils the proof in the non-unital case is the fact that operators of the form $a = a_1 \otimes 1 + 1 \otimes a_2$ are in general not elements of $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$.

Lemma 9. *Let $a \in \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ and for any two $\varphi_i \in S(\mathcal{A}_i)$, $i = 1, 2$, let us call $a_1 := (\text{id} \otimes \varphi_2)(a) \in \mathcal{A}_1$ and $a_2 := (\varphi_1 \otimes \text{id})(a) \in \mathcal{A}_2$. Then*

$$\|[D_1, a_1]\|_{\mathcal{B}(\mathcal{H}_1)} \leq \|[D_1 \otimes 1, a]\|_{\mathcal{B}(\mathcal{H})} \quad \text{and} \quad \|[D_2, a_2]\|_{\mathcal{B}(\mathcal{H}_2)} \leq \|[\gamma_1 \otimes D_2, a]\|_{\mathcal{B}(\mathcal{H})} .$$

Proof. We use the obvious identification of $\mathcal{A}_1 \otimes \mathbb{C} 1$ with \mathcal{A}_1 and $\mathbb{C} 1 \otimes \mathcal{A}_2$ with \mathcal{A}_2 . We also identify $\mathcal{B}(\mathcal{H}_1) \otimes \mathbb{C} 1$ with $\mathcal{B}(\mathcal{H}_1)$ and $\mathbb{C} 1 \otimes \mathcal{B}(\mathcal{H}_2)$ with $\mathcal{B}(\mathcal{H}_2)$.

States have norm 1. Hence the maps $\text{id} \otimes \varphi_2 : \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{A}_2 \rightarrow \mathcal{B}(\mathcal{H}_1)$ and $\varphi_1 \otimes \text{id} : \mathcal{A}_1 \otimes \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_2)$ have norm 1. This means

$$\begin{aligned} \|[D_1, a_1]\|_{\mathcal{B}(\mathcal{H}_1)} &= \|[D_1, (\text{id} \otimes \varphi_2)(a)]\|_{\mathcal{B}(\mathcal{H}_1)} \\ &= \|(\text{id} \otimes \varphi_2)[D_1 \otimes 1, a]\|_{\mathcal{B}(\mathcal{H}_1)} \leq \|[D_1 \otimes 1, a]\|_{\mathcal{B}(\mathcal{H})} . \end{aligned}$$

Similarly, one has $\|[D_2, a_2]\|_{\mathcal{B}(\mathcal{H}_2)} \leq \|[1 \otimes D_2, a]\|_{\mathcal{B}(\mathcal{H})}$. To conclude the proof we notice that, since $\gamma_1^* \gamma_1 = 1$, $\|[1 \otimes D_2, a]\|_{\mathcal{B}(\mathcal{H})} = \|(\gamma_1 \otimes 1)[1 \otimes D_2, a]\|_{\mathcal{B}(\mathcal{H})} = \|[\gamma_1 \otimes D_2, a]\|_{\mathcal{B}(\mathcal{H})}$. ■

Lemma 10. *Let γ be a grading, A an odd operator and B an even operator (i.e. $A\gamma + \gamma A = 0$ and $B\gamma - \gamma B = 0$). Then*

$$\max \{ \|A\|_{\mathcal{B}(\mathcal{H})}, \|B\|_{\mathcal{B}(\mathcal{H})} \} \leq \|A + B\|_{\mathcal{B}(\mathcal{H})} . \quad (17)$$

Proof. From the triangle inequality we obtain

$$\|A\|_{\mathcal{B}(\mathcal{H})} = \left\| \frac{A+B}{2} + \frac{A-B}{2} \right\|_{\mathcal{B}(\mathcal{H})} \leq \frac{1}{2} \|A + B\|_{\mathcal{B}(\mathcal{H})} + \frac{1}{2} \|A - B\|_{\mathcal{B}(\mathcal{H})} ,$$

and

$$\|B\|_{\mathcal{B}(\mathcal{H})} = \left\| \frac{A+B}{2} - \frac{A-B}{2} \right\|_{\mathcal{B}(\mathcal{H})} \leq \frac{1}{2} \|A + B\|_{\mathcal{B}(\mathcal{H})} + \frac{1}{2} \|A - B\|_{\mathcal{B}(\mathcal{H})} .$$

But $A - B = -\gamma(A + B)\gamma$ with $\gamma = \gamma^*$ unitary, so $\|A - B\|_{\mathcal{B}(\mathcal{H})} = \|A + B\|_{\mathcal{B}(\mathcal{H})}$. This proves that $\|A\|_{\mathcal{B}(\mathcal{H})} \leq \|A + B\|_{\mathcal{B}(\mathcal{H})}$ and $\|B\|_{\mathcal{B}(\mathcal{H})} \leq \|A + B\|_{\mathcal{B}(\mathcal{H})}$, i.e. the inequality (17). ■

Corollary 11. *For any $a \in \mathcal{A}$, we have*

$$\max \left\{ \|[D_1 \otimes 1, a]\|_{\mathcal{B}(\mathcal{H})}, \|[\gamma_1 \otimes D_2, a]\|_{\mathcal{B}(\mathcal{H})} \right\} \leq \|[D, a]\|_{\mathcal{B}(\mathcal{H})}^2. \quad (18)$$

Proof. Apply Lemma 10 with $A = [D_1 \otimes 1, a]$, $B = [\gamma_1 \otimes D_2, a]$ and $\gamma = \gamma_1 \otimes 1$. \blacksquare

Proof of Theorem 5, point (i). Let $\varphi = \varphi_1 \otimes \varphi_2$ and $\varphi' = \varphi'_1 \otimes \varphi'_2$ be two separable states. Any $a \in \mathcal{A}$ can be written as $a = \sum_i a_1^i \otimes a_2^i$. Notice that

$$\begin{aligned} \varphi(a) - \varphi'(a) &= \sum_i \varphi_1(a_1^i) \varphi_2(a_2^i) - \varphi'_1(a_1^i) \varphi'_2(a_2^i) \\ &= \sum_i \left\{ \varphi_1(a_1^i) - \varphi'_1(a_1^i) \right\} \varphi_2(a_2^i) + \varphi'_1(a_1^i) \left\{ \varphi_2(a_2^i) - \varphi'_2(a_2^i) \right\} \\ &= \varphi_1(a_1) - \varphi'_1(a_1) + \varphi_2(a_2) - \varphi'_2(a_2) \\ &\leq d_1(\varphi_1, \varphi'_1) \|[D_1, a_1]\|_{\mathcal{B}(\mathcal{H}_1)} + d_2(\varphi_2, \varphi'_2) \|[D_2, a_2]\|_{\mathcal{B}(\mathcal{H}_2)}, \end{aligned}$$

where we used the linearity of states and called

$$a_1 = \sum_i a_1^i \varphi_2(a_2^i) \in \mathcal{A}_1, \quad a_2 = \sum_i a_2^i \varphi'_1(a_1^i) \in \mathcal{A}_2.$$

Using Lemma 9 we deduce that

$$\varphi(a) - \varphi'(a) \leq d_1(\varphi_1, \varphi'_1) \|[D_1 \otimes 1, a]\|_{\mathcal{B}(\mathcal{H})} + d_2(\varphi_2, \varphi'_2) \|[\gamma_1 \otimes D_2, a]\|_{\mathcal{B}(\mathcal{H})}.$$

By (18) we get

$$\varphi(a) - \varphi'(a) \leq \left\{ d_1(\varphi_1, \varphi'_1) + d_2(\varphi_2, \varphi'_2) \right\} \|[D, a]\|_{\mathcal{B}(\mathcal{H})},$$

and taking the sup over \mathcal{A} with $\|[D, a]\|_{\mathcal{B}(\mathcal{H})} \leq 1$ we get (14a). \blacksquare

We remark that, unlike (14b), (14a) and (15) are valid for arbitrary (not necessarily unital) spectral triples. In §6.1 and §6.2 we give two counterexamples to (14b) using non-unital spectral triples. In the next section we make a short digression to explain the importance of this counterexamples, arising in the study of K-homology.

5 Interlude on the one-point and two-point spaces

5.1 K-homology of \mathbb{C}

An even pre-Fredholm module $(\mathcal{A}, \mathcal{H}, F, \gamma)$ over a $*$ -algebra \mathcal{A} is given by a \mathbb{Z}_2 -graded Hilbert space \mathcal{H} with grading γ , a representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, commuting with the grading, and a bounded operator F anticommuting with the grading, such that $\pi(a)(F - F^*)$, $\pi(a)(F^2 - 1)$ and $[F, \pi(a)]$ are compact operators for all $a \in \mathcal{A}^2$. With a suitable equivalence relation, classes of even pre-Fredholm modules form the zeroth K-homology group $K^0(\mathcal{A})$, see e.g. [16, §8.2].

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$, a pre-Fredholm module $(\mathcal{A}, \mathcal{H}, F, \gamma)$ can be obtained by replacing D with $F = D(1 + D^2)^{-\frac{1}{2}}$. Vice versa, any K-homology class has a representative that arises from a spectral triple through this exact construction [1].

²Here we adopt the terminology of [14, §8.2]. In [1, 16] pre-Fredholm modules are called Fredholm modules ‘tout court’.

We recall that in any K-homology class one can find a representative such that $F = F^*$ and $F^2 = 1$: this will be called a Fredholm module [14, §8.2]. A Fredholm module is called 1-summable if $[F, \pi(a)]$ is of trace class for all $a \in \mathcal{A}$ [5].

Clearly, if \mathcal{H} is finite-dimensional, any Fredholm module $(\mathcal{A}, \mathcal{H}, F, \gamma)$ is 1-summable and it is also a spectral triple (the resolvent and bounded commutator conditions are trivially satisfied). There is a pairing between K-homology and K-theory, that for 1-summable even Fredholm modules is given by

$$\langle [F], [p] \rangle = \frac{1}{2} \text{Tr}_{\mathcal{H} \otimes \mathbb{C}^n} (\gamma F [F, \pi(p)]) , \quad (19)$$

where: $p = p^2 = p^* \in M_n(\mathcal{A})$ is a projection, representative of an element $[p]$ in the K-theory group $K_0(\mathcal{A})$, $[F]$ is the class of the 1-summable Fredholm module $(\mathcal{A}, \mathcal{H}, F, \gamma)$, π is extended to a representation of $M_n(\mathcal{A})$ on $\mathcal{H} \otimes \mathbb{C}^n$ in the obvious way, and $\text{Tr}_{\mathcal{H} \otimes \mathbb{C}^n}$ is the trace on $\mathcal{H} \otimes \mathbb{C}^n$. Using (19) any Fredholm module corresponds to a linear map:

$$\text{ch}_F : K_0(\mathcal{A}) \rightarrow \mathbb{C} , \quad \text{ch}_F([p]) := \langle [F], [p] \rangle , \quad (20)$$

usually called Chern-Connes character.

Suppose we are interested in Fredholm modules for the algebra \mathbb{C} , i.e. functions on the space with one point. For any $z \in \mathbb{C}$ one has $\pi(z) = z\pi(1)$, and if the representation is unital, then $\pi(1)$ commutes with any operator F , so that the Chern-Connes character (20) is identically zero. To get a non-zero Chern-Connes character, we need to use a representation π that is not unital.

A non-trivial Fredholm module $(\mathbb{C}, \widetilde{\mathcal{H}}, \widetilde{F}, \widetilde{\gamma})$ is given by $\widetilde{\mathcal{H}} = \mathbb{C}^2$, with representation, operator \widetilde{F} and grading given by:

$$\widetilde{\pi}(z) = \begin{bmatrix} z & 0 \\ 0 & 0 \end{bmatrix} , \quad \widetilde{F} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \widetilde{\gamma} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

Note that

$$\text{Tr}_{\mathbb{C}^2} (\widetilde{\gamma} \widetilde{F} [\widetilde{F}, \widetilde{\pi}(z)]) = 2z .$$

Given an element $[p] \in K_0(\mathbb{C})$, from (19) we get

$$\langle [\widetilde{F}], [p] \rangle = \frac{1}{2} \text{Tr}_{\mathbb{C}^2} (\widetilde{\gamma} \widetilde{F} [\widetilde{F}, \sum_i \widetilde{\pi}(p_{ii})]) = \sum_i p_{ii} .$$

This is exactly the rank of p . It is well known that the above Fredholm module generates $K^0(\mathbb{C}) \simeq \mathbb{Z}$ (any other Fredholm module is equivalent to a multiple of this one).

5.2 Pull-back of Fredholm modules/“amplification” of spectral triples

Here we describe two ways to construct Fredholm modules or spectral triples on an algebra \mathcal{A} using the basic Fredholm module $(\mathbb{C}, \widetilde{\mathcal{H}}, \widetilde{F}, \widetilde{\gamma})$ of previous section. These will be applied then to the study of the algebra \mathbb{C}^2 .

5.2.1 Pull-back

Given a connected locally compact Hausdorff space X , we can use an irreducible representation — i.e. a map $C_0(X) \rightarrow \mathbb{C}$, $f \mapsto f(x)$, with $x \in X$ — to obtain a Fredholm module over $C_0(X)$ as a pullback of the Fredholm module over \mathbb{C} given above: the corresponding Chern-Connes map, evaluated on a projection, gives the rank of the corresponding vector bundle. This is true even for some quantum spaces, for example quantum complex projective spaces [9]. For X compact, one of the generators of $K_0(C(X))$ is the trivial projection $p = 1$ (the constant function); to get a full set of generators of $K^0(C(X))$ one is forced to use the construction above, as any Fredholm module with a unital representation will have a trivial pairing with $p = 1$.

More generally if \mathcal{A} is any associative involutive complex algebra, one can use a one-dimensional irreducible representation $\chi : \mathcal{A} \rightarrow \mathbb{C}$ (if any) to pull-back the Fredholm module $(\mathbb{C}, \widetilde{\mathcal{H}}, \widetilde{F}, \widetilde{\gamma})$. The result is a Fredholm module $(\mathcal{A}, \widetilde{\mathcal{H}}, \widetilde{F}, \widetilde{\gamma})$ with $\widetilde{\mathcal{H}}$, \widetilde{F} , and $\widetilde{\gamma}$ as above, and with representation of \mathcal{A} on $\widetilde{\mathcal{H}}$ given by $\widetilde{\pi} \circ \chi$:

$$\widetilde{\pi} \circ \chi(a) = \begin{bmatrix} \chi(a) & 0 \\ 0 & 0 \end{bmatrix}, \quad \forall a \in \mathcal{A}.$$

This is what we do, for example, for quantum complex projective spaces or for the standard Podleś sphere, to get the last generator of the K-homology [9].

5.2.2 Amplification

As explained, the Fredholm module $(\mathbb{C}, \widetilde{\mathcal{H}}, \widetilde{F}, \widetilde{\gamma})$ above is also an even spectral triple (over the space with one point), and given any other spectral triple $(\mathcal{A}, \pi_0, \mathcal{H}_0, D_0)$ we can form their product, that we denote by $(\mathcal{A}, \pi, \mathcal{H}, D)$ (we explicitly indicate the representation symbols). Clearly we are not changing the algebra: $\mathcal{A} \otimes \mathbb{C} \simeq \mathcal{A}$. On the other hand, $\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}^2$, and the representation and Dirac operator are given by:

$$\pi(a) = \begin{bmatrix} \pi_0(a) & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} D_0 & 1 \\ 1 & -D_0 \end{bmatrix}.$$

If the former spectral triple is even, with grading γ_0 , the latter is even too, with grading

$$\gamma = \begin{bmatrix} \gamma_0 & 0 \\ 0 & -\gamma_0 \end{bmatrix}.$$

The advantage is that one can start from a spectral triple that is trivial in K-homology, and get a new spectral triple with a non-trivial K-homology class.

5.3 K-homology of \mathbb{C}^2

The K-theory and K-homology of $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$ are well known (here we denote elements as pairs (a, b) , instead of writing $a \oplus b$). We know that the K-theory is $K_0(\mathbb{C} \oplus \mathbb{C}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ (see e.g. Exercise 6.I(h) of [26]), with generators given by

$$p_+ := (1, 0), \quad p_- := (0, 1).$$

(We are not interested in K_1 , that in this example is zero anyway.)

Concerning K-homology, using the two irreducible representations of \mathbb{C}^2 , given by the two pure states $\varphi_+(a, b) = a$ and $\varphi_-(a, b) = b$, we can get two Fredholm modules $(\mathbb{C}^2, \mathcal{H}_+, F_+, \gamma_+)$ and $(\mathbb{C}^2, \mathcal{H}_-, F_-, \gamma_-)$, as explained in §5.2.1. We have $\mathcal{H}_+ = \mathcal{H}_- = \mathbb{C}^2$,

$$F_+ = F_- = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma_+ = \gamma_- = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and the only difference is in the representation

$$\pi_+(a, b) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \quad \pi_-(a, b) = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}.$$

Using (20) one checks that the pairing with K-theory is $\langle [F_i], [p_j] \rangle = \delta_{ij}$, proving that we have a dual pair of generators of K-theory and K-homology.

For metric purposes, these two Fredholm modules are not very interesting since the corresponding spectral distance between pure states is infinite (in both cases we have elements not proportional to 1 commuting with the ‘Dirac’ operator F : those in the kernel of the representation). We now describe two spectral triples that are more suitable for metric purposes.

The first unital spectral triple $(\mathcal{A}_1, \pi_1, \mathcal{H}_1, D_1, \gamma_1)$ is given by $\mathcal{A}_1 = \mathcal{H}_1 = \mathbb{C}^2$, with

$$\pi_1(a, b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad D_1 = \frac{1}{\lambda} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

with $\lambda > 0$ a fixed scale. The distance between the two pure states of the algebra is easily computed, and given by (see e.g. page 35 of [5]):

$$d_{\mathcal{A}_1, D_1}(\varphi_0, \varphi_1) = \lambda. \quad (21)$$

The corresponding Fredholm module is given by $F_1 = \lambda D_1$.

Another even Fredholm module $(\mathcal{A}_2, \pi_2, \mathcal{H}_2, F_2, \gamma_2)$ is obtained as the ‘‘amplification’’ of $(\mathcal{A}_1, \pi_1, \mathcal{H}_1, 0, I_2)$ (note that the grading I_2 anticommutes with the zero Dirac operator), as explained in §5.2.2. The result is $\mathcal{H}_2 = \mathbb{C}^4$, with grading $\gamma_2 = \text{diag}(1, 1, -1, -1)$, and with (non-unital) $*$ -representation $\pi_2 : \mathbb{C}^2 \rightarrow M_4(\mathbb{C})$ and F_2 given by:

$$\pi_2(a, b) = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

For the Dirac operator, it is convenient to choose the normalization $D_2 = 2\mu^{-1}F_2$, with $\mu > 0$ a fixed length scale. One easily checks that $\|[D_2, \pi_2(a, b)]\| = 2\mu^{-1} \max\{|a|, |b|\}$. From this, it follows that

$$d_{\mathcal{A}_2, D_2}(\varphi_0, \varphi_1) = \mu. \quad (22)$$

The Dirac operators D_1 and D_2 correspond to geometries that are ‘‘topologically’’ inequivalent, i.e. to different classes in $K^0(\mathbb{C}^2)$. More precisely, computing the pairing with the projections p_+ and p_- one proves the following relations with the generators of $K^0(\mathbb{C}^2)$:

$$[F_1] = [F_+] - [F_-], \quad [F_2] = [F_+] + [F_-].$$

6 The importance of being non-degenerate

The proof of (14b) works only for unital spectral triples. In §6.1 and §6.2 we show what happens if one of the two spectral triples is not unital: in the former case the algebra is unital but the representation is not, in the latter case the algebra is itself non-unital. In both cases the inequality (14b) is not true (it is violated already by pure states).

6.1 A product of two-point spaces

Here we consider the the product $(\mathcal{A}, \pi, \mathcal{H}, D)$ of the spectral triples $(\mathcal{A}_1, \pi_1, \mathcal{H}_1, D_1, \gamma_1)$ and $(\mathcal{A}_2, \pi_2, \mathcal{H}_2, D_2)$ over the algebra $\mathcal{A}_1 = \mathcal{A}_2 = \mathbb{C}^2$ introduced in §5.3. From (21) and (22) we have:

$$d_{\mathcal{A}_1, D_1}(\varphi_+, \varphi_-) = \lambda, \quad d_{\mathcal{A}_2, D_2}(\varphi_+, \varphi_-) = \mu,$$

where $\varphi_+(a, b) = a$ and $\varphi_-(a, b) = b$ are the two pure states of \mathbb{C}^2 . If (14a) were true in the non-unital case, we would expect $d_{\mathcal{A}, D}(\varphi_+ \otimes \varphi_+, \varphi_- \otimes \varphi_-) \geq \sqrt{\lambda^2 + \mu^2}$. The next proposition shows that this is not the case.

Proposition 12. *The distance $d_{\mathcal{A}, D}(\varphi_+ \otimes \varphi_+, \varphi_- \otimes \varphi_-) = \mu$ is independent of λ .*

Proof. Recall that $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $D = D_1 \otimes 1 + \gamma_1 \otimes D_2$. If $\{e_i\}_{i=1}^n$ is the canonical orthonormal basis of \mathbb{C}^n , a unitary map $U : \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^4 \rightarrow \mathbb{C}^8$ is defined by $U(e_1 \otimes e_i) = e_i$ and $U(e_2 \otimes e_i) = e_{i+4} \forall i = 1, \dots, 4$.

An isomorphism $\iota : \mathcal{A} = \mathbb{C}^2 \otimes \mathbb{C}^2 \rightarrow \mathbb{C}^4$ is given by

$$\iota((a_1, a_2) \otimes (b_1, b_2)) := (a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2).$$

The states $\varphi_+ \otimes \varphi_+$ and $\varphi_- \otimes \varphi_-$ can be pulled-back to states on \mathbb{C}^4 given by

$$(\varphi_+ \otimes \varphi_+) \circ \iota^{-1}(a_1, \dots, a_4) = a_1, \quad (\varphi_- \otimes \varphi_-) \circ \iota^{-1}(a_1, \dots, a_4) = a_4.$$

The representation $\pi_1 \otimes \pi_2$ gives the following representation $\pi(a) := U((\pi_1 \otimes \pi_2)\iota^{-1}(a))U^*$ of $a = (a_1, \dots, a_4) \in \mathbb{C}^4$ that is explicitly given by:

$$\pi(a_1, \dots, a_4) = \text{diag}(a_1, a_2, 0, 0, a_3, a_4, 0, 0),$$

and the Dirac operator becomes the 8×8 matrix

$$D = \begin{bmatrix} D_2 & \lambda^{-1} I_4 \\ \lambda^{-1} I_4 & -D_2 \end{bmatrix},$$

where I_4 is the 4×4 identity matrix. The distance $d_{\mathcal{A}, D}(\varphi_+ \otimes \varphi_+, \varphi_- \otimes \varphi_-)$ is then the supremum of $a_1 - a_4$ over $a = (a_1, \dots, a_4) \in \mathbb{R}^4$ (it is enough to consider self-adjoint elements) with the condition $\|[D, \pi(a)]\| \leq 1$. Applying the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 7 & 8 & 3 & 4 & 5 & 6 \end{pmatrix}$ to rows and columns of $[D, \pi(a)]$, one gets the matrix

$$\begin{bmatrix} 0_4 & -B_a \\ B_a^* & 0_4 \end{bmatrix}, \quad \text{with} \quad B_a := \frac{1}{\mu} \begin{bmatrix} 2a_1 & 0 & \mu\lambda^{-1}(a_1 - a_3) & 0 \\ 0 & 2a_2 & 0 & \mu\lambda^{-1}(a_2 - a_4) \\ 0 & 0 & 2a_3 & 0 \\ 0 & 0 & 0 & 2a_4 \end{bmatrix}.$$

Since permutation matrices are unitary, $\|[D, \pi(a)]\| = \|B_a\|$. Denoting by b_{ij} the matrix elements of B_a , since $b_{ij} \leq \|B_a\|$, we have

$$a_1 - a_4 = \frac{\mu}{2}(b_{11} - b_{44}) \leq \mu \|B_a\| = \mu \|[D, \pi(a)]\| ,$$

proving that the distance is no greater than μ . If $a_1 = -a_2 = a_3 = -a_4 = \mu/2$, then $B_a = \text{diag}(1, -1, 1, -1)$ has norm 1, so that the distance is no less than $a_1 - a_4 = \mu$. ■

Corollary 13. *For $\lambda/\mu \geq 0$ ($\mu \neq 0$), the ratio*

$$k_\lambda := \frac{d_{\mathcal{A},D}(\varphi_+ \otimes \varphi_+, \varphi_- \otimes \varphi_-)}{\sqrt{d_{\mathcal{A}_1,D_1}(\varphi_+, \varphi_-)^2 + d_{\mathcal{A}_2,D_2}(\varphi_+, \varphi_-)^2}} = \frac{\mu}{\sqrt{\lambda^2 + \mu^2}}$$

assumes all the values in the interval $(0, 1]$ (compare this with the situation in §3.3).

Similarly to (15), we could think of replacing (14b) with a weaker inequality

$$d(\varphi, \varphi') \geq k \sqrt{d_1(\varphi_1, \varphi'_1)^2 + d_2(\varphi_2, \varphi'_2)^2} , \quad (23)$$

for some $k \geq 0$. Last corollary shows that the only value of k such that (23) is valid for all spectral triples is $k = 0$.

Observe also that Cor. 6 is no longer valid in the non-unital case. What spoils the proof is the “ \geq ” inequality. Since what really matter is the ratio λ/μ , from now on $\mu = 1$.

Proposition 14. *If $\lambda > 1$ (and $\mu = 1$), then $d_{\mathcal{A},D}(\varphi_+ \otimes \varphi_+, \varphi_- \otimes \varphi_+) < d_{\mathcal{A}_1,D_1}(\varphi_+, \varphi_-)$.*

Proof. The distance between $\varphi_+ \otimes \varphi_+$ and $\varphi_- \otimes \varphi_+$ is the supremum of $a_1 - a_3$ (over $a_1, \dots, a_4 \in \mathbb{R}$) with the constraint that $\|B_a\| \leq 1$. Note that

$$a_1 - a_3 = \frac{\lambda}{1+\lambda} \{(a_1 - a_3) + \lambda^{-1}(a_1 - a_3)\} \leq \frac{\lambda}{1+\lambda} \{|a_1| + |a_3| + \lambda^{-1}|a_1 - a_3|\} . \quad (24)$$

Every $n \times n$ matrix B satisfies (cf. equations (2.3.11) and (2.3.12) of [13]):

$$\|B\|_\infty := \max_{1 \leq i \leq n} \sum_{j=1}^n |b_{ij}| \leq \sqrt{n} \|B\| \quad \text{and} \quad \|B\|_1 := \max_{1 \leq j \leq n} \sum_{i=1}^n |b_{ij}| \leq \sqrt{n} \|B\| .$$

In our case $n = 4$, and looking at the third column resp. first row we get:

$$2|a_3| + \lambda^{-1}|a_1 - a_3| \leq \|B_a\|_\infty \leq 2\|B_a\| , \quad 2|a_1| + \lambda^{-1}|a_1 - a_3| \leq \|B_a\|_1 \leq 2\|B_a\| .$$

Thus $|a_1| + |a_3| + \lambda^{-1}|a_1 - a_3| \leq 2\|B_a\|$ and by (24): $a_1 - a_3 \leq \frac{2\lambda}{1+\lambda} \|B_a\|$. This proves that $d_{\mathcal{A},D}(\varphi_+ \otimes \varphi_+, \varphi_- \otimes \varphi_+) \leq \frac{2\lambda}{1+\lambda}$. On the other hand $d_{\mathcal{A}_1,D_1}(\varphi_+, \varphi_-) = \lambda$, and

$$\frac{2\lambda}{1+\lambda} < \lambda$$

for every $\lambda > 1$. This concludes the proof. ■

Remark 15. *For $\lambda \rightarrow \infty$, $d_{\mathcal{A}_1,D_1}(\varphi_+, \varphi_-)$ diverges while $d_{\mathcal{A},D}(\varphi_+ \otimes \varphi_+, \varphi_- \otimes \varphi_+) \leq 2$.*

6.2 Two-sheeted real line

In this example, the first (unital) spectral triple $(\mathcal{A}_1, \pi_1, \mathcal{H}_1, D_1, \gamma_1)$ is the one introduced in §5.3, depending on a scale $\lambda > 0$. The second (non-unital) spectral triple $(\mathcal{A}_2, \pi_2, \mathcal{H}_2, F_2)$ is obtained as the “amplification”, cf. §5.2.2, of the canonical spectral triple of the real line:

$$(C_0^\infty(\mathbb{R}), L^2(\mathbb{R}), \tfrac{1}{2}\mathcal{D}), \quad \mathcal{D} := i \frac{d}{dx}.$$

(The normalization $1/2$ of the Dirac operator allows to simplify some formulas.) The result is $\mathcal{A}_2 = C_0^\infty(\mathbb{R})$ with representation π_2 on $\mathcal{H}_2 = L^2(\mathbb{R}) \otimes \mathbb{C}^2$ Dirac operator F_2 given by:

$$\pi_2(f) = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}, \quad F_2 = \frac{1}{2} \begin{bmatrix} \mathcal{D} & 2 \\ 2 & -\mathcal{D} \end{bmatrix}.$$

where f acts on $L^2(\mathbb{R})$ by pointwise multiplication. To get nicer formulas, we prefer to compute the distance using $D_2 = 2F_2$ rather than F_2 .

Now, let $(\mathcal{A}, \mathcal{H}, D)$ be the product of $(\mathcal{A}_1, \pi_1, \mathcal{H}_1, D_1, \gamma_1)$ and $(\mathcal{A}_2, \pi_2, \mathcal{H}_2, D_2)$.

Proposition 16. *For any $x, y \in \mathbb{R}$ and $\lambda > 0$, we have*

$$d_{\mathcal{A}, D}(\varphi_+ \otimes \delta_x, \varphi_- \otimes \delta_y) \leq 1 \leq \lambda^{-1} \sqrt{d_{\mathcal{A}_1, D_1}(\varphi_+, \varphi_-)^2 + d_{\mathcal{A}_2, D_2}(\delta_x, \delta_y)^2}.$$

Notice that for $\lambda \rightarrow \infty$, $d_{\mathcal{A}_1, D_1}(\varphi_+, \varphi_-)$ diverges while $d_{\mathcal{A}, D}(\varphi_+ \otimes \delta_x, \varphi_- \otimes \delta_x)$ is never greater than 1. In particular, if $\lambda > 1$ one has

$$d_{\mathcal{A}, D}(\varphi_+ \otimes \delta_x, \varphi_- \otimes \delta_x) \neq d_{\mathcal{A}_1, D_1}(\varphi_+, \varphi_-).$$

Proof. Recall that $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $D = D_1 \otimes 1 + \gamma_1 \otimes D_2$.

We identify $\mathbb{C}^2 \otimes \mathcal{H}_2$ with $L^2(\mathbb{R}) \otimes \mathbb{C}^4$. The representation of $(f_+, f_-) \in \mathbb{C}^2 \otimes C_0(\mathbb{R})$ is $\pi(f_+, f_-) = \text{diag}(f_+, 0, f_-, 0)$, and the Dirac operator is

$$D = \begin{bmatrix} D_2 & \lambda^{-1} I_2 \\ \lambda^{-1} I_2 & -D_2 \end{bmatrix} = \begin{bmatrix} \mathcal{D} & 2 & \lambda^{-1} & 0 \\ 2 & -\mathcal{D} & 0 & \lambda^{-1} \\ \lambda^{-1} & 0 & -\mathcal{D} & -2 \\ 0 & \lambda^{-1} & -2 & \mathcal{D} \end{bmatrix},$$

where I_2 is the 2×2 identity matrix. We have

$$[D, \pi(f_+, f_-)] = \begin{bmatrix} if'_+ & -2f_+ & -\lambda^{-1}(f_+ - f_-) & 0 \\ 2f_+ & 0 & 0 & 0 \\ \lambda^{-1}(f_+ - f_-) & 0 & -if'_- & 2f_- \\ 0 & 0 & -2f_- & 0 \end{bmatrix}.$$

By taking the sup over vectors with only the second component different from zero we prove that $\|2f_+\|_\infty \leq \|[D, \pi(f_+, f_-)]\|$. Similarly $\|2f_-\|_\infty \leq \|[D, \pi(f_+, f_-)]\|$. But

$$(\varphi_+ \otimes \delta_x)(f_+, f_-) - (\varphi_- \otimes \delta_y)(f_+, f_-) = f_+(x) - f_-(y) \leq \|f_+\|_\infty + \|f_-\|_\infty \leq \|[D, \pi(f_+, f_-)]\|,$$

and this proves that $d_{\mathcal{A}, D}(\varphi_+ \otimes \delta_x, \varphi_- \otimes \delta_y) \leq 1$. This proves the first inequality in Prop. 16. The other one follows from the simple observation that

$$d_{\mathcal{A}_1, D_1}(\varphi_+, \varphi_-)^2 + d_{\mathcal{A}_2, D_2}(\delta_x, \delta_y)^2 \geq d_{\mathcal{A}_1, D_1}(\varphi_+, \varphi_-)^2 = \lambda^2,$$

last equality being (21). ■

7 Conclusion

As often advertised by Connes, the “line element” in noncommutative geometry has to be thought as the inverse of the Dirac operator,

$$“\, ds \sim D^{-1} \, ”.$$

For a product of two spectral triples $X_1 = (\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $X_2 = (\mathcal{A}_2, \mathcal{H}_2, D_2)$, noticing that

$$D^2 = (D_1 \otimes \Gamma_2 + 1 \otimes D_1)^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2, \quad (25)$$

this yields a “inverse Pythagoras equality” [8]

$$\frac{1}{ds^2} = \frac{1}{ds_1^2} + \frac{1}{ds_2^2}. \quad (26)$$

In [22], we discussed why it was possible to invert (26) in case of the product of a manifold by \mathbb{C}^2 , so that to retrieve Pythagoras theorem. The main result of this paper is to show that for the product of arbitrary (unital) spectral triples, (26) can be “integrated” and leads to the inequalities (6) for the spectral distance. More precisely, if $\varphi := \varphi_1 \otimes \varphi_2$ and $\varphi' = \varphi'_1 \otimes \varphi'_2$ are arbitrary separable states on $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ (not necessarily pure), and d (resp. d_i) is the spectral distance of X (resp. X_i) then the inequalities (6) hold:

$$\sqrt{d_1(\varphi_1, \varphi'_1)^2 + d_2(\varphi_2, \varphi'_2)^2} \leq d(\varphi, \varphi') \leq \sqrt{2} \sqrt{d_1(\varphi_1, \varphi'_1)^2 + d_2(\varphi_2, \varphi'_2)^2}.$$

These inequalities already appeared in [20, Prop. II.4], where one of the two spaces was assumed to be the two-point space \mathbb{C}^2 and only pure states were considered. Here we proved them in full generality: (6b) holds for arbitrary spectral triples and (6a) holds if the spectral triples X_1 and X_2 are both unital.

In §4 we provide two elementary (commutative) examples, where one of the two spectral triples is non-unital, and we show that the inequality (6a) is violated even by pure states.

In the case of the Wasserstein distance, we argued that Pythagoras equality is a property of pure states, and it does not hold if we consider non-pure states. Besides the product of two manifolds, it is not clear whether the purity of states is an essential condition to retrieve Pythagoras theorem: for the product of a manifold by \mathbb{C}^2 , the distance between two separable non-pure states $\varphi = \varphi_1 \otimes \varphi_2$, $\varphi' = \varphi'_1 \otimes \varphi'_2$ is still unknown, except when $\varphi_i = \varphi'_i$ for either $i = 1$ or $i = 2$. Then Pythagoras equality is trivially satisfied. For the product of the Moyal plane by \mathbb{C}^2 , the purity of the states does not seem to be a relevant criterion: Pythagoras theorem holds true for translated states, pure or not, and we do not know whether it holds for arbitrary pure states. In any case, it would be interesting to find a nice class of noncommutative spectral triples where Pythagoras equality can be proved for pure states.

References

- [1] S. Baaj and P. Julg, *Théorie bivariante de Kasparov et opérateurs non bornés dans les C^* -modules hilbertiens*, C. R. Acad. S. Paris 296 (1983) 875–878.

- [2] A. H. Chamseddine, A. Connes, and M. Marcolli. Gravity and the standard model with neutrino mixing. *Adv. Theor. Math. Phys.*, 11:991–1089, 2007.
- [3] A. Connes, *Noncommutative differential geometry*, Inst. Hautes Etudes Sci. Publ. Math. 62 (1985), 257–360.
- [4] A. Connes, *Compact metric spaces, Fredholm modules, and hyperfiniteness*, Ergodic Theory Dynam. Systems 9 (1989), 207–220.
- [5] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.
- [6] A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. 36 (1995) 6194–6231.
- [7] A. Connes, *On the spectral characterization of manifolds*, arXiv:0810.2088v1 [math.OA].
- [8] A. Connes, *Variations sur le thème spectral*, summary of Collège de France lectures (2007) available at http://www.college-de-france.fr/media/alain-connes/UPL53971_2.pdf; see also <http://noncommutativegeometry.blogspot.com/search?q=harmonic+mean>.
- [9] F. D’Andrea and G. Landi, *Bounded and unbounded Fredholm modules for quantum projective spaces*, J. K-theory 6 (2010) 231–240.
- [10] F. D’Andrea and P. Martinetti, *A view on Transport Theory from Noncommutative Geometry*, SIGMA 6 (2010) 057.
- [11] L. Dabrowski and G. Dossena, *Product of real spectral triples*, Int. J. Geom. Methods Mod. Phys. 8 (2011) 1833–1848.
- [12] P.B. Gilkey, *Invariance theory, the heat equation and the Atiyah-Singer index theorem*, electronic reprint, 1996 (printed version published by Publish or Perish Press, 1984).
- [13] G.H. Golub and C.F. van Loan, *Matrix Computations*, 3rd ed., Johns Hopkins Univ. Press, 1996.
- [14] J.M. Gracia-Bondía, J.C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser, 2001.
- [15] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs of the AMS 16, AMS, 1979.
- [16] N. Higson and J. Roe, *Analytic K-homology*, Oxford Univ. Press, 2000
- [17] B. Iochum, T. Krajewski, and P. Martinetti. *Distances in finite spaces from noncommutative geometry*, J. Geom. Phys., 31 (2001) 100–125.
- [18] R. V. Kadison and J. R. Ringrose. *Fundamentals of the Theory of Operator Algebras. Volume II, Advanced theory*, volume IV. Academic Press, 1986.
- [19] P. Martinetti, F. Mercati and L. Tomassini, *Minimal length in quantum space and integrations of the line element in Noncommutative Geometry*, Rev. Math. Phys. 24 (2012) 1250010-36 pages.
- [20] P. Martinetti and L. Tomassini, *Noncommutative geometry of the Moyal plane: translation isometries and spectral distance between coherent states*, arXiv:1110.6164v1 [math-ph].
- [21] P. Martinetti and R. Wulkenhaar, *Discrete Kaluza-Klein from scalar fluctuations in noncommutative geometry*, J. Math. Phys. 43 1 (2002) 182–204.
- [22] P. Martinetti, *Line element in quantum gravity: the examples of dsr and noncommutative geometry*, Int. J. Mod. Phys. A, 24 (15) (2009) 2792–2801.
- [23] M. A. Rieffel, *Metric on state spaces*, Documenta Math., 4 (1999) 559–600.
- [24] F.-J. Vanhecke, *On the product of real spectral triples*, Lett. Math. Phys. 50 (1999) 157–162.
- [25] C. Villani, *Optimal Transport: Old and New*, Grundlehren der mathematischen Wissenschaften, vol. 338, Springer, 2009.
- [26] N.E. Wegge-Olsen, *K-theory and C*-algebras. A friendly approach*, Oxford, 1993.